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NONLINEAR OSCILLATIONS IN EQUATIONS WITH DELAYS.(U)
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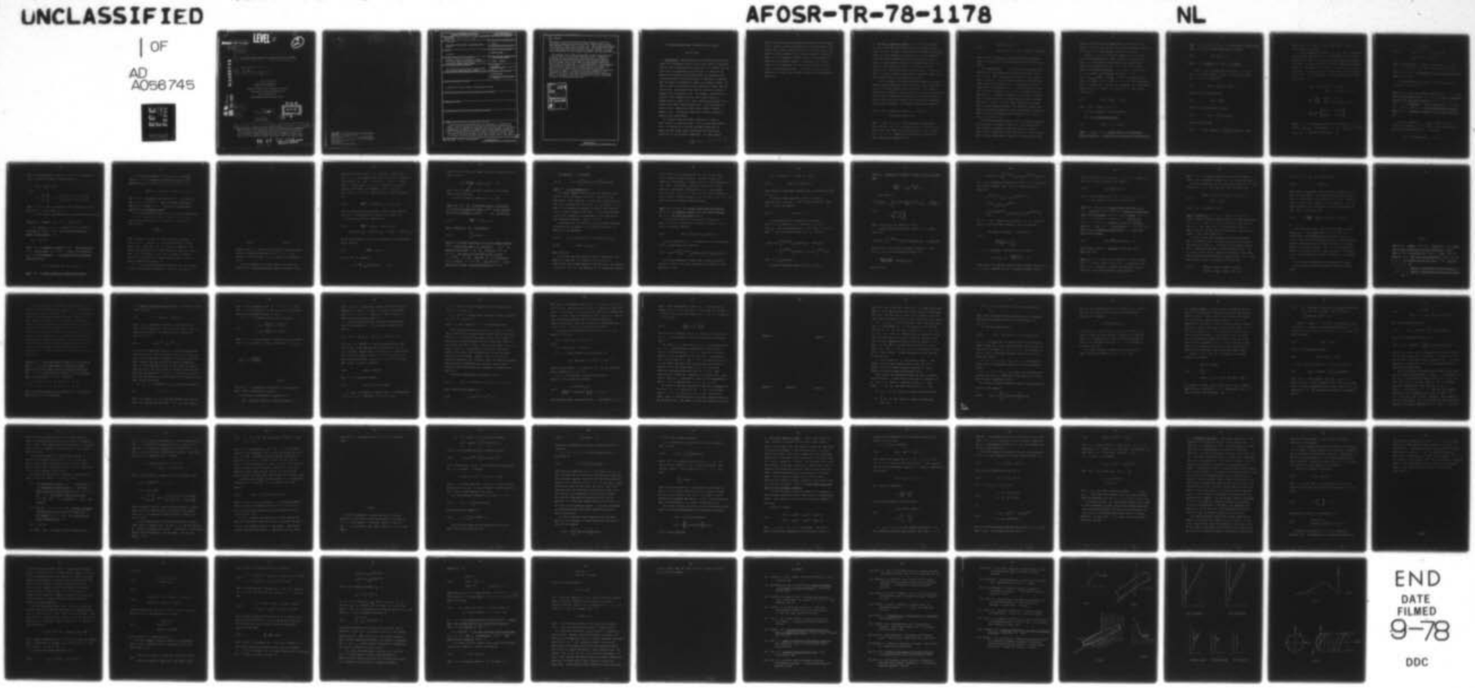

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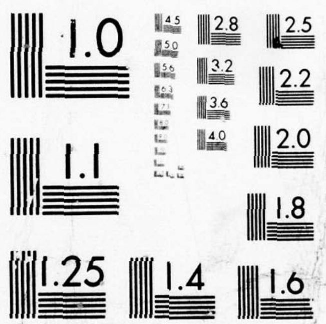
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These lectures are concerned only with some aspects of bifurcation theory in the local theory of nonlinear oscillations in equations with delays; that is, behavior of solutions near an equilibrium. In particular, ~~we study~~ how the qualitative behavior of solutions change is shown as parameters vary. A detailed study of the local theory is important in order to know the types of solutions to expect in a global problem. Of course, there is no reason to only study local theory near an equilibrium. One should next page

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20. Abstract

study how the qualitative behavior changes near any invariant set - for example, behavior near a periodic orbit, behavior near an orbit which connects a saddle point to itself, etc. More complicated behavior is expected near these large invariant sets. One can obtain invariant torii, homoclines points which exhibit a chaotic behavior, etc. We restrict ourselves in these lectures to behavior near equilibrium.

The simplest type of smooth bifurcation is from an equilibrium to a periodic orbit - the so-called Hopf bifurcation. In Section 2, we discuss the Hopf bifurcation in equations with finite delays permitting the bifurcation parameters to be the delays themselves. At first glance, such a result does not seem possible because the vector field in the equation is not differentiable in the parameters. The theorem does require some new ideas and, for this reason, the proof is given in some detain. Several examples are given in Section 3. In Section 4, similar results are presented for equations with infinite delays. In Section 5, we give an example in two dimensions for which stable Hopf bifurcation occurs with decreasing delay. In Section 6, we give an introduction to some of the methods available for nonautonomous equations.

NONLINEAR OSCILLATIONS IN EQUATIONS WITH DELAYS

Jack K. Hale

1. Introduction. These lectures are concerned only with some aspects of bifurcation theory in the local theory of nonlinear oscillations in equations with delays;that is, behavior of solutions near an equilibrium. In particular, we study how the qualitative behavior of solutions change as parameters vary. A detailed study of the local theory is important in order to know the types of solutions to expect in a global problem. Of course, there is no reason to only study local theory near an equilibrium. One should study how the qualitative behavior changes near any invariant set - for example, behavior near a periodic orbit, behavior near an orbit which connects a saddle point to itself, etc. More complicated behavior is expected near these large invariant sets. One can obtain invariant torii, homoclines points which exhibit a chaotic behavior, etc. We restrict ourselves in these lectures to behavior near equilibrium.

The simplest type of smooth bifurcation is from an equilibrium to a periodic orbit - the so-called Hopf bifurcation. In Section 2, we discuss the Hopf bifurcation in equations with finite delays permitting the bifurcation parameters to be the delays themselves. At first glance,

such a result does not seem possible because the vector field in the equation is not differentiable in the parameters. The theorem does require some new ideas and, for this reason, the proof is given in some detail. Several examples are given in Section 3. In Section 4, similar results are presented for equations with infinite delays. In Section 5, we give an example in two dimensions for which stable Hopf bifurcation occurs with decreasing delay. In Section 6, we give an introduction to some of the methods available for nonautonomous equations.

2. The Hopf bifurcation theorem. One of the simplest ways in which nonconstant periodic solutions of autonomous equations can arise is when an equilibrium point changes from being stable to being unstable as parameters vary in the equation. A periodic orbit can bifurcate from the equilibrium, the process being generally referred to as Hopf bifurcation. For ordinary differential equations, several proofs of the existence of a Hopf bifurcation have been given. In some way, they all involve ultimately the implicit function theorem and the technicalities of the proofs are minimal.

In the statement of the Hopf bifurcation theorem, one must always impose differentiability conditions on the vector field in order to obtain smooth bifurcation curves. For ordinary differential equations, these conditions are not very restrictive. For functional differential equations, the obvious differentiability conditions can eliminate the discussion of variations in important parameters. For example, consider the equation

$$\begin{aligned}(2.1)\qquad \dot{x}(t) &= A(\delta)x(t) + B(\delta)x(t-r) + C(\delta)x(t-s)\\ &\quad + f(\delta,x(t),x(t-r),x(t-s))\end{aligned}$$

where $\delta \in \mathbb{R}$, $r > 0$, $s > 0$ are considered as parameters, $A(\delta),B(\delta),C(\delta)$ are $n \times n$ constant matrices, $f(\delta,x,y,z)$ as well as the first derivatives with respect to x,y,z vanish at $x = y = z = 0$. Suppose the characteristic equation of the linear part of Equation (2.1)

$$\det \Delta(\lambda,\delta,r,s) = 0 \tag{2.2}$$

$$\Delta(\lambda,\delta,r,s) = \lambda I - A(\delta) - B(\delta)e^{-\lambda r} - C(\delta)e^{-\lambda s}$$

for $(\delta,r,s) = (\delta_0,r_0,s_0)$ has a pair of purely imaginary roots $i\nu_0, -i\nu_0, \nu_0 > 0$, and all other roots have negative real parts.

The fundamental pr[illegible] is to discuss the existence and stability of small nonconstant periodic solutions of (2.1) for (δ,r,s) near (δ_0,r_0,s_0) and which vary in a smooth way in $(\delta,r,s,)$. It is not very restrictive to assume the right hand side of Equation (2.1) is continuous and has a continuous first derivative in δ. One can then prove that the solution of Equation (2.1) is also continuously differentiable in δ. However, regardless of the space of initial data for Equation (2.1), the solution map will not be differentiable in (r,s). At first sight, this makes it unclear how to solve the above problem. The important feature that makes the problem tractable is that every periodic solution of Equation (2.1) must have one more derivative in t than $f(\delta,x,y,z)$ has in x,y,z.

The Hopf bifurcation theorem is not an elementary exercise for functional differential equations. The verification of the hypotheses necessary to apply the implicit function theorem uses a number of special identities for linear systems with constant coefficients. Since we are going to state the

theorem so that it will be applicable to variations in the delays, a proof of the theorem will be indicated and is based on the proof of a less general result in [7, p. 246].

Suppose $r \geq 0$ is a given real number, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^n$ is an n-dimensional linear vector space over the reals with norm $|\cdot|$, $C([a,b],\mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. If $[a,b] = [-r,0]$, we let $C = C([-r,0],\mathbb{R}^n)$ and designate the norm of ϕ by $|\phi| = \sup_{-r \leq \theta \leq 0} |\phi(\theta)|$. Suppose $\Omega \subset \mathbb{R}^k$ is an open set (the parameter space), $f\colon \Omega \times C \to \mathbb{R}^n$, $L\colon \Omega \times C \to \mathbb{R}^n$ are continuous, $L(\alpha)\phi$ is linear in ϕ, $f(\alpha,\phi)$ has continuous first and second derivatives in ϕ, $f(\alpha,0) = 0$, $\partial f(\alpha,0)/\partial\phi = 0$ and consider the equation

$$\dot{x}(t) = L(\alpha)x_t + f(\alpha, x_t), \tag{2.3}$$

where $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$.

Our first hypothesis is the following:

(H_1) The characteristic matrix,

$$\Delta(\alpha,\lambda) = \lambda I - L(\alpha)e^{\lambda \cdot} I, \tag{2.4}$$

where I is the $n \times n$ identity matrix, is continuously differentiable in α, there is a purely imaginary characteristic

<u>root</u> $\lambda_0 = i\nu_0$, $\nu_0 > 0$, <u>for</u> $\alpha = \alpha_0$ <u>and no other characteristic root</u> $\lambda_j \neq \lambda_0, \bar{\lambda}_0$ <u>of the characteristic equation</u>

$$\det \Lambda(\alpha,\lambda) = 0 \tag{2.5}$$

<u>for</u> $\alpha = \alpha_0$ <u>satisfies</u> $\lambda_j = m\lambda_0$, m <u>an integer.</u>

Since $L(\alpha,\phi)$ is continuous and linear in ϕ, there is an $n \times n$ matrix function $\eta(\alpha,\theta)$ of bounded variation in θ, $-r \leq \theta \leq 0$, such that

$$L(\alpha)\phi = \int_{-r}^{0} [d\eta(\alpha,\theta)]\phi(\theta). \tag{2.6}$$

Along with the linear equation

$$\dot{x}(t) = L(\alpha)x_t \tag{2.7}$$

consider the formal adjoint equation

$$\dot{y}(\tau) = -\int_{-r}^{0} y(\tau-\theta)d\eta(\alpha,\theta) \tag{2.8}$$

and the bilinear form

$$(\psi,\phi) = \psi(0)\phi(0) - \int_{-r}^{0}\int_{0}^{\theta}\psi(\xi-\theta)[d\eta(\alpha,\theta)]\phi(\xi)d\xi \tag{2.9}$$

defined for $\phi \in C$ and $\psi \in C([0,r],\mathbb{R}^{n^*})$, $\mathbb{R}^{n^*}$ = space of n-dimensional row vectors.

For the characteristic values $(i\nu_0, -i\nu_0)$ of the characteristic equation $\det \Delta(\alpha_0,\lambda) = 0$ corresponding to the linear equation (2.7) for $\alpha = \alpha_0$, there are two linearly independent solutions $b \cos \nu_0 t$, $b \sin \nu_0 t$ for some n-dimensional column vector b. In the same way, there are two linearly independent solutions $c \cos \nu_0 \tau$, $c \sin \nu_0 \tau$ of the formal adjoint equation (2.8) for $\alpha = \alpha_0$. In Chapter 7 of [7], the following remarks are proved. If

$$\Phi_{\alpha_0} = (\phi_1,\phi_2),\quad \phi_1(\theta) = b \cos \nu_0\theta$$
$$\phi_2(\theta) = b \sin \nu_0\theta,\quad -r \le \theta \le 0$$

$$\Psi^*_{\alpha_0} = \begin{pmatrix} \psi^*_1 \\ \psi^*_2 \end{pmatrix},\quad \begin{aligned} \psi^*_1(s) &= c \cos \nu_0 s \\ \psi^*_2(s) &= c \sin \nu_0 s,\quad 0 \le s \le r \end{aligned}$$

$$(\Psi^*_{\alpha_0},\Phi^*_{\alpha_0}) = \begin{bmatrix} (\psi_1,\phi_1) & (\psi_1,\phi_2) \\ (\psi_2,\phi_1) & (\psi_2,\phi_1) \end{bmatrix}$$

where (,) is the bilinear form in (2.9), then the 2×2 matrix $(\Psi^*_{\alpha_0},\Phi_{\alpha_0})$ is nonsingular. If $\Psi_{\alpha_0} = (\Psi^*_{\alpha_0},\Phi_{\alpha_0})^{-1}\Psi^*_\alpha$, then $(\Psi_{\alpha_0},\Phi_{\alpha_0}) = I$ and one can decompose C as

$$C = P_{\alpha_0} \oplus Q_{\alpha_0},$$

$$(2.10) \qquad P_{\alpha_0} = \{\phi \in C: \phi = \Phi_{\alpha_0} a,\ a \in \mathbb{R}^2\}$$

$$Q_{\alpha_0} = \{\phi \in C: (\Psi_{\alpha_0}, \phi) = 0\}$$

This decomposition of C defines a projection π_{α_0} with $\pi_{\alpha_0} C = P_{\alpha_0}$, $(I - \pi_{\alpha_0})C = Q_{\alpha_0}$. When a decomposition is made in this way, we say C is decomposed by the set of characteristic values $\{i\nu_0, -i\nu_0\}$.

The following result is proved in exactly the same way as Lemma 2.2 in [7, p. 171].

Lemma 2.1. If (H_1) is satisfied, then there is a $\delta > 0$ and a simple characteristic root $\lambda(\alpha)$ of (2.5) which is continuous together with its first derivative, $\operatorname{Im} \lambda(\alpha) > 0$, $|\alpha - \alpha_0| < \delta$, $\lambda(\alpha_0) = i\nu_0$. Furthermore, C can be decomposed by $\{\lambda(\alpha), \overline{\lambda}(\alpha)\}$ as $C = P_\alpha \oplus Q_\alpha$, $\dim P_\alpha = 2$, and the corresponding projection operator π_α is continuous together with its first derivative in α.

With this Lemma and $\Phi_\alpha = (\phi_1, \phi_2)$ a basis for P_α, it follows that there is a 2×2 matrix $B(\alpha)$ continuous and continuously differentiable in α such that

$$\Phi_\alpha(\theta) = \Phi_\alpha(0) \exp B(\alpha)\theta, \quad -r \leq \theta \leq 0.$$

Also, the eigenvalues of $B(\alpha)$ are $\lambda(\alpha), \bar{\lambda}(\alpha)$. Furthermore, we may assume by a change of coordinates that

$$B(\alpha) = \nu_0 B_0 + B_1(\alpha)$$

(2.11)

$$B_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} (\alpha-\alpha_0)\cdot\zeta(\alpha) & (\alpha-\alpha_0)\cdot\gamma(\alpha) \\ -(\alpha-\alpha_0)\cdot\gamma(\alpha) & (\alpha-\alpha_1)\cdot\zeta(\alpha) \end{bmatrix}$$

where $\zeta(\alpha), \gamma(\alpha) \in \mathbb{R}^k$ are continuous and continuously differentiable in α for $|\alpha-\alpha_0| < \delta$.

We may now state a generalization of the Hopf bifurcation theore

Theorem 2.1. *Suppose* $L(\alpha)\phi$, $f(\alpha,\phi)$ *satisfy* (H_1),

(H_2) *for any* $K > 0$, $\phi \in C$ *with* $d\phi/d\theta \in C$, $|d\phi/d\theta| \leq K$, *the function* $L(\alpha)\phi + f(\alpha,\phi)$ *has a first derivative in* α *which is continuous in* α, ϕ,

(H_3) $\zeta(\alpha_0) \neq 0$

where $\zeta(\alpha)$ *is defined in Equation* (2.11). *Then there is an* $\varepsilon > 0$ *such that for* $a \in \mathbb{R}$, $|a| < \varepsilon$, *there is a* C^1*-manifold* $\Gamma_a \in \mathbb{R}_k$ *of codimension* 1, Γ_a *is continuous and continuously differential in a*

$$\Gamma_0 = \{\alpha \in \mathbb{R}^k : \text{Re}\,\lambda(\alpha) = 0,\ |\alpha-\alpha_0| < \varepsilon\}$$

where $\lambda(\alpha)$ *is given in Lemma 2.1, such that for every*

$\alpha \in \Gamma_a$, there is a function $\omega(\alpha,a)$, an $\omega(\alpha,a)$-periodic function $x^*(\alpha,a)$ continuous together with their first derivatives in a,α, $\omega(\alpha_0,0) = \omega_0 = 2\pi/\nu_0$, $x^*(\alpha,0) = 0$,

$$x_0^*(\alpha,a) = \Phi_\alpha \operatorname{col}(a,0) + o(|a|) \quad \text{as} \quad |a| \to 0$$

and $x^*(\alpha,a)$ is a solution of Equation (2.3). Furthermore, for $|\alpha-\alpha_0| < \varepsilon$, $|\omega-\omega_0| < \varepsilon$ every ω-periodic solution of Equation (2.3) with $|x_t| < \varepsilon$ must be of the above type except for a translation in phase.

Before proving the theorem, let us discuss some implications.

If $k = 1$, that is, $\alpha \in \mathbb{R}$, and $\lambda(\alpha) = \mu(\alpha) + i\nu(\alpha)$ is the characteristic root given by Lemma 2.1, then (H_3) is equivalent to

$$\frac{d\lambda(\alpha_0)}{d\alpha} \neq 0.$$

Thus, Theorem 2.1 is the usual Hopf bifurcation theorem. In fact, for $k = 1$, the set Γ_a can be described by a smooth curve $\alpha(a)$. Thus, for $|a| < \varepsilon$, there is a function $\alpha(a)$ continuous together with its first derivative such that $\alpha(0) = \alpha_0$ and $x^*(a) = x^*(\alpha(a),a)$ is an $\omega(a) = \omega(\alpha(a),a)$ periodic solution of Equation (2.3). The bifurcation curve $\omega(a)$ is shown schematically in Figure 1.

If $k = 2$, then the bifurcation surface in the (a,α)-space is shown schematically in Figure 2. For each a, there is a

Figure 1 Figure 2

smooth curve of α values for which there exist nonconstant periodic solutions of Equation (2.3). If $k > 2$, the picture is similar except Γ_a is a surface of codimension 1 in $\mathbb{R}^k$.

To prove Theorem 2.1, we first obtain the bifurcation equations for the small amplitude periodic solutions of the

equation (2.3) for $|\alpha-\alpha_0| < \delta$ which have a period close to $2\pi/\nu_0$. As in ordinary differential equations, we normalize the period to 2π. Let $\beta \in (-1,1)$, $\omega_0 = 2\pi/\nu_0$, $t = (1+\beta)\tau$, $x(t+\theta) = u(\tau+\theta/(1+\beta))$, $-r \leq \theta \leq 0$ and define $u_{\tau,\beta}$ as an element of the space $C([-r,0],\mathbb{R}^n)$ given by $u_{\tau,\beta}(\theta) = u(\tau+\theta/(1+\beta))$, $-r \leq \theta \leq 0$. Equation (2.3) is then equivalent to

$$\frac{du(\tau)}{d\tau} = (1+\beta)[L(\alpha)u_{\tau,\beta} + f(\alpha,u_{\tau,\beta})]. \tag{2.12}$$

If this equation has an ω_0-periodic solution, then Equation (2.3) has a $(1+\beta)\omega_0$-periodic solution, and conversely.

Equation (2.12) may be rewritten as

$$\frac{du(\tau)}{d\tau} = L(\alpha_0)u_\tau + N(\beta,\alpha,u_\tau,u_{\tau,\beta}) \tag{2.13}$$

$$N(\beta,\alpha,u_\tau,u_{\tau,\beta}) = (1+\beta)L(\alpha)u_{\tau,\beta} - L(\alpha_0)u_\tau + (1+\beta)f(\alpha,u_{\tau,\beta}).$$

We consider Equation (2.13) as a perturbation of the autonomous linear equation

$$\frac{du(\tau)}{d\tau} = L(\alpha_0)u_\tau . \tag{2.14}$$

We know that the columns of

$$U(\tau) \overset{\text{def}}{=} \Phi_{\alpha_0}(0)\exp B(\alpha_0)\tau, \quad \tau \in \mathbb{R},$$

form a basis for the ω_0-periodic solutions of Equation (2.14) and the rows of

$$V(\tau) \stackrel{\text{def}}{=} [\exp - B(\alpha_0)\tau]\Psi_{\alpha_0}(0), \quad \tau \in \mathbb{R}$$

form a basis for the ω_0-periodic solutions of the formal adjoint Equation (2.8).

We need the following lemma from [7, p. 209].

Lemma 2.2. Let $\mathscr{P}_{\omega_0}$ be the Banach space of continuous ω_0-periodic functions with values in $\mathbb{R}^n$ and the topology of uniform convergence. For any $f \in \mathscr{P}_{\omega_0}$, the equation

$$\frac{du(\tau)}{d\tau} = L(\alpha_0)u_\tau + f(\tau) \tag{2.15}$$

has a solution in $\mathscr{P}_{\omega_0}$ if and only if

$$\int_0^{\omega_0} y(t)f(t)dt = 0 \tag{2.16}$$

for all ω_0-periodic solutions of the formal adjoint equation (2.8). Furthermore, there is a projection $J: \mathscr{P}_{\omega_0} \to \mathscr{P}_{\omega_0}$ such that the set of f in $\mathscr{P}_{\omega_0}$ satisfying (2.16) is $(I-J)\mathscr{P}_{\omega_0}$ and there is a continuous linear operator $K: (I-J)\mathscr{P}_{\omega_0} \to (I-\tilde{\pi})\mathscr{P}_{\omega}$ such that Kf is a solution of (2.15) for every $f \in (I-J)\mathscr{P}_{\omega_0}$. The operator $\tilde{\pi}$ is any projection onto the ω_0-periodic solutions of (2.14).

The operator J is given by

$$(2.17) \qquad Jf = V'\left[\int_0^{\omega_0} V(s)V'(s)ds\right]^{-1}\int_0^{\omega_0} V(s)f(s)ds$$

where V' is the transpose of V.

This lemma simply states necessary and sufficient conditions for the existence of periodic solutions of nonhomogeneous linear equations. Knowing these conditions, one can always subtract off part of any $f \in \mathscr{P}_{\omega_0}$, namely Jf, so that the orthogonality conditions (2.16) are satisfied for $(I-J)f$. Thus, a solution will exist. We can always choose a particular solution to be orthogonal to the solutions of the homogeneous equation. This is $K(I-J)f$.

Using Lemma 2.2, we know that every ω_0-periodic solution of Equation (2.13) except for a translation in phase is a solution of the equations

$$(2.18a) \qquad u(\cdot) = U(\cdot)\mathrm{col}(a,o) + K(I-J)N(\beta,\alpha,u_{\cdot},u_{\cdot,\beta})$$

$$(2.18b) \qquad JN(\beta,\alpha,u_{\cdot},u_{\cdot,\beta}) = 0$$

and conversely.

One can now apply the implicit function theorem to solve Equation (2.18a) for $u = u^*(a,\beta,\alpha)$ for a,β,α in a sufficiently small neighborhood of zero, $u^*(a,0,0) - U(\cdot)\mathrm{col}(a,0) = o(|a|)$ as $|a| \to 0$. The function u^* is continuous together

with derivatives up through order two from an application of the implicit function theorem. However, without some additional information, we cannot prove it is differentiable in β,α. Since $u^*(a,\beta,\alpha)(t)$ satisfies (2.18a), it is automatically continuously differentiable in t with bounded derivative for $|\alpha-\alpha_0| < \delta$. If we use this fact and the form of the implicit function theorem given in Lemma 2.2 in [7, p. 236], one obtains the following important result.

Lemma 2.3. *If* (H_1), (H_2) *are satisfied, then the function* $u^*(a,\beta,\alpha)$ *is continuous together with its first derivatives in* a,β,α *and second derivatives in* a.

From the above lemma, all ω_0-periodic solutions of Equation (2.13) are obtained by finding the solutions a,β,α of the bifurcation equations

$$(2.19) \qquad JN(\beta,\alpha,u^*_{\cdot}(a,\beta,\alpha),u^*_{\cdot,\beta}(a,\beta,\alpha)) = 0.$$

Using the definition of J in Equation (2.17) the Equation (2.19) is equivalent to the equation

$$(2.20) \qquad G(a,\beta,\alpha) \overset{\text{def}}{=} \int_0^{\omega_0} e^{-B(\alpha_0)s}\,\Psi_{\alpha_0}(0)N(\beta,\alpha,u^*_s(a,\beta,\alpha),u^*_{s,\beta}(a,\beta,\alpha))ds = 0$$

From the above discussion, it follows that it remains to solve the equation $G(a,\beta,\alpha) = 0$, which represents two equations in the parameters a,β,α.

Since $G(0,\beta,\alpha) = 0$ for all β,α, define

$$H(a,\beta,\alpha) = G(a,\beta,\alpha)/a. \tag{2.21}$$

This function is continuous together with its first derivatives in a,β,α.

The nonzero small amplitude periodic solutions of Equation (2.3) are given as $x(t) = u^*(a,\beta,\alpha)(t/1+\beta))$ where a,β,α satisfy the equation

$$H(a,\beta,\alpha) = 0. \tag{2.22}$$

Our next objective is to compute the Jacobian of $H(a,\beta,\alpha)$ with respect to β,α evaluated at $(a,\beta,\alpha) = (0,0,\alpha_0)$. From the definition of G in (2.20), N in (2.13) and u^* one easily observes that, for $e_1 = \mathrm{col}(1,0)$,

$$\begin{aligned} H(0,0,\alpha) &= \int_0^{\omega_0} e^{-B(\alpha_0)s}\Psi_{\alpha_0}(0)\{L(\alpha)U_s e_1 - L(0)U_s e_1\}ds \\ &= \int_0^{\omega_0} e^{-B(\alpha_0)s}\Psi_{\alpha_0}(0)[L(\alpha) - L(\alpha_0)]\Phi_{\alpha_0} e^{B(\alpha_0)s} e_1 ds \end{aligned}$$

since $U_s = \Phi_{\alpha_0}\exp B(\alpha_0)s$.

We need the following Lemma 3.9 in [7, p. 179].

Lemma 2.4. Suppose the conditions of Lemma 2.1 are satisfied. Then

$$\frac{dB(\alpha_0)}{d\alpha} = -\Psi_{\alpha_0}(0)\,\frac{dL(\alpha_0)}{d\alpha}\,\Phi_{\alpha_0}.$$

Using Lemma 2.4,

$$\frac{\partial H(0,0,\alpha_0)}{\partial\alpha}\,\nu = \int_0^{\omega_0} e^{-B(\alpha_0)s}\,\frac{dB(\alpha_0)}{d\alpha}\,e^{B(\alpha_0)s}e_1\,ds = \omega_0\,\frac{dB(\alpha_0)}{d\alpha}\,e_1$$

(2.23)

$$= \omega_0\begin{bmatrix} \nu\cdot\zeta(\alpha_0) \\ -\nu\cdot\gamma(\alpha_0) \end{bmatrix}$$

where $\zeta(\alpha_0), \gamma(\alpha_0)$ are defined in (2.11).

To compute the derivative with respect to β, we note first that

$$H(0,\beta,\alpha_0) = \int_0^{\omega_0} e^{-B(\alpha_0)s}\,\Psi_{\alpha_0}(0)\{(1+\beta)L(\alpha_0)U_{s,\beta}e_1 - L(\alpha_0)U_s e_1\}ds.$$

Writing this as two separate integrals, changing s into $s/(1+\beta)$ in the first integral and noting that

$$\frac{dU(s/(1+\beta))}{d(s/(1+\beta))} = L(\alpha_0)U_{s/(1+\beta),\beta}$$

one sees that

$$H(0,\beta,\alpha_0) = \beta \int_0^{\omega_0} e^{-B(\alpha_0)s} \Psi_{\alpha_0}(0)\, \Phi_{\alpha_0}(0) e^{B(\alpha_0)s} B(\alpha_0) e_1 ds.$$

If x is a solution of Equation (2.14) and y is a solution of the adjoint equation, then (y^t, x_t) = constant for all t. Therefore,

$$I = (e^{-B(\alpha_0)(s+\cdot)} \Psi_{\alpha_0}(0),\ \Phi_{\alpha_0}(0) e^{B(\alpha_0)(s+\cdot)})$$

$$= e^{-B(\alpha_0)s} \Psi_{\alpha_0}(0) \Phi_{\alpha_0}(0) e^{B(\alpha_0)s} -$$

$$\int_{-r}^{0} \int_0^{\theta} e^{-B(\alpha_0)(s+\xi-\theta)} \Psi_{\alpha_0}(0) d\eta(\alpha_0,\theta) \Phi_{\alpha_0}(0) e^{B(\alpha_0)(s+\xi)} d\xi$$

for all $s \in \mathbb{R}$. Integrating this from 0 to ω_0 and using the fact that the second integral is zero, one obtains

$$H(0,\beta,\alpha_0) = \beta\omega_0 B(\alpha_0) e_1 = -2\pi\beta \operatorname{col}(0,1)$$

(2.24)

$$\frac{\partial H(a,\beta,\alpha_0)}{\partial \beta} = \begin{bmatrix} 0 \\ -2\pi \end{bmatrix}.$$

If $H = \operatorname{col}(H_1, H_2)$ then Relation (2.24) implies

$$H\ (0,0,\alpha_0) = 0, \quad \frac{\partial H_2(0,0,\alpha_0)}{\partial \beta} = -2\pi.$$

Consequently, the implicit function theorem implies there is an $\varepsilon > 0$ and a function $\beta^*(\alpha, a)$ continuous together with its

first derivatives in $\alpha, a, |\alpha-\alpha_0| < \varepsilon$, $|a| < \varepsilon$, $\beta^*(\alpha_0,0) = 0$, such that $\beta^*(\alpha,a)$ satisfies the equation

(2.25) $$H_2(a,\beta^*(\alpha,a),\alpha) = 0$$

and is the only solution for $|\beta| < \varepsilon$, $|\alpha-\alpha_0| < \varepsilon$, $|a| < \varepsilon$.

Up to this point, we have the following lemma.

Lemma 2.5. *If* (H_1), (H_2) *are satisfied, then there is an* $\varepsilon > 0$, *functions* $\beta^*(\alpha,a)$, $u^*(a,\beta,\alpha)$, *continuous together with their first derivatives*, $u^*(a,\beta,\alpha)$ *is* ω_0*-periodic*, $|a| < \varepsilon$, $|\alpha-\alpha_0| < \varepsilon$, $\beta^*(\alpha_0,0) = 0$, $u^*(0,0,\alpha_0) = 0$, *such that for* $|\omega-\omega_0| < \varepsilon$, $|\alpha-\alpha_0| < \varepsilon_0$, *Equation (2.3) has an* ω*-periodic solution* x *with* $|x| < \varepsilon$ *if and only if* $x = u^*(a,\beta^*(\alpha,a),\alpha)$ *except for a translation in phase where* α, a *satisfy the bifurcation equation*

(2.26) $$h(\alpha,a) \overset{\text{def}}{=} H_1(a,\beta^*(\alpha,a),\alpha) = 0.$$

The function $u^*(a,\beta,\alpha)$ *satisfies* (2.18a) *and* $\beta^*(\alpha,a)$ *satisfies* (2.5).

Remark 2.1. It is clear from the proof of the above results that if we assume $L(\alpha)\phi + f(\alpha,\phi)$ has k derivatives with respect to ϕ which are continuous and Hypothesis (H_2) is satisfied for the k^{th} derivatives with respect to α

when ϕ has k continuous derivatives, then the function $h(\alpha,a)$ has continuous derivatives in α,a up through order k.

From the expressions for the partial derivatives of H with respect to α,β at $(0,0,\alpha_0)$, one easily determines that

$$\begin{aligned} h(\alpha,a) &= \xi_0(a) + (\alpha-\alpha_0)\xi_1(a) + o(|\alpha-\alpha_0|) \\ \xi_1(0) &= \zeta(\alpha_0), \end{aligned} \tag{2.27}$$

where $\zeta(\alpha_0)$ is given in (2.11).

Proof of Theorem 2.1. If $\Gamma_a = \{\alpha: h(\alpha,a) = 0, |a| < \varepsilon\}$, then the Implicit Function Theorem implies the assertions about Γ_a if $k = 1$. For $k > 1$, it is the transversality theorem (see [1, p. 45]). Then $\omega(\alpha,a) = (1+\beta^*(\omega,a))\omega_0$, $x^*(\alpha,a) = u^*(a,\beta^*(\alpha,a),\alpha)$ satisfy all of the properties stated in the theorem. This proves Theorem 2.1.

Theorem 2.1 uses only knowledge of the linear operator $L(\alpha)$ in Equation (2.3). We know nothing about the specific structure of the sets Γ_α. To obtain a more complete picture of the bifurcation, one must consider nonlinear terms. Let us suppose $L(\alpha)\phi$, $f(\alpha,\phi)$ satisfy all conditions above with the differentiability conditions up through order $k \geq 3$. Then the function $h(\alpha,a)$ in Relation (2.27) satisfies

$$\begin{aligned} h(\alpha,a) &= h(\alpha,0) + p(\alpha)a^2 + o(|a|^2), \\ h(\alpha,0) &= (\alpha-\alpha_0) + \zeta(\alpha_0) + o(|\alpha-\alpha_0|), \end{aligned} \tag{2.28}$$

as $|a| \to 0$, $\alpha \to \alpha_0$. Let us assume that

$$p(\alpha_0) \neq 0. \tag{2.29}$$

The function $h(\alpha,a)/p(\alpha)$ will then have a minimum at $a = 0$ given by $h(\alpha,0)/p(\alpha)$. The bifurcation equation will have a solution if and only if this minimum is less than or equal to zero and will have no solution if this minimum is > 0. The minimum equal to zero is therefore a bifurcation surface in the α-space. This is given by

$$0 = \frac{h(\alpha,0)}{p(\alpha)} = \frac{1}{p(\alpha_0)} (\alpha-\alpha_0)\cdot\zeta(\alpha_0) + o(|\alpha-\alpha_0|) \tag{2.30}$$

as $\alpha \to \alpha_0$.

Under hypotheses (H_3), Equation (2.30) defines a smooth hyperplane Q in a neighborhood of α_0 in $\mathbb{R}^k$. In a sufficiently small neighborhood of the point $(\alpha_0,0)$ in $\Omega \times C$, on one side of this hyperplane there are no nonconstant periodic solutions and on the other side there is a unique nonconstant periodic solution. For $k = 2$, the bifurcation diagram is similar to the one shown in Figure 3. The curve is the intersection of the surface in Figure 3 with the plane $a = 0$.

The direction of bifurcation is determined by $p(\alpha_0)$.

These results are briefly summarized in the following result.

Figure 3

Theorem 2.2. Suppose $L(\alpha)\phi, f(\alpha,\phi)$ satisfy (H_1), (H_2) taking into account derivatives up through order three. If, in addition, (H_3) is satisfied and $p(\alpha_0) \neq 0$ where $p(\alpha)$ is given in (2.28), then there are neighborhoods $V \subset \mathbb{R}^k$ of α_0, U in C of zero and a smooth hyperplane $\Gamma \subset \mathbb{R}^k$ such that $V \backslash \Gamma = A_1 \cup A_2$ and

(i) $\alpha \in A_1$ implies no nonconstant periodic solution in U

(ii) $\alpha \in A_2$ implies a unique nonconstant periodic solution in U.

A less general version of the Existence Theorem 2.2 was obtained by Ruiz-Claeyssen [22, Theorem 6.2]. He considered the equation (2.3) in the space $W^{1,\infty}$ in order to obtain the center manifold and then applied techniques similar to the ones employed by Chafee [4]. In addition, he was able to extend the analysis in Hale [9], Hausrath [11] to obtain the stability of the bifurcation when (H_1) was strengthened to say that all characteristic values have negative real parts except the two on the imaginary axis. The stability properties are determined by the change in stability of the zero solution as Γ is considered. One can actually improve the results in [22] as well as extend them to the case when Equation (2.3) is considered in C. More specifically, one can prove the following result.

<u>Theorem 2.3.</u> <u>If the conditions of Theorem 2.2 are satisfied and for</u> $\alpha = \alpha_0$, <u>all eigenvalues of Equation (2.7) have negative real parts except two on the imaginary axis, then for</u> $\alpha \in V$ <u>all eigenvalues of Equation (2.7) have negative real parts except two,</u> $\lambda(\alpha), \bar{\lambda}(\alpha)$ <u>and either</u>

(i) $\operatorname{Re} \lambda(\alpha) < 0$ in A_1, $\operatorname{Re} \lambda(\alpha) > 0$ in A_2

or

(ii) $\operatorname{Re} \lambda(\alpha) > 0$ in A_1, $\operatorname{Re} \lambda(\alpha) < 0$ in A_2.

<u>The bifurcating periodic orbit is stable if (i) is satisfied and unstable if (ii) is satisfied.</u>

3. <u>Examples of variations in the delays</u>. Consider the linear scalar equation

$$(3.1) \qquad \dot{x} = -\frac{1}{2}x(t-r_1) - \frac{1}{2}x(t-r_2)$$

where r_1, r_2 are positive constants. The specific coefficients in Equation (3.1) are chosen in order to make the computations simpler. The characteristic equation for Equation (3.1) is

$$(3.2) \qquad \lambda + \frac{1}{2}e^{-\lambda r_1} + \frac{1}{2}e^{-\lambda r_2} = 0.$$

To disuuss the Hopf bifurcation for a nonlinear perturbation of Equation (3.1), one must determine the curves in (r_1, r_2)-space at which the roots of Equation 3.2 have zero real parts. Ruiz-Claeyssen [22] has analyzed this problem completely. There are infinitely many curves for which there are purely imaginary roots, but we are going to concentrate on the curve which has the property if $(r_1, r_2) \in \Gamma$, then there is a pair of purely imaginary roots and all other roots of Equation (3.2) have negative real parts.

For $r_1 = r_2 = r$, Equation (3.2) is the familiar equation

$$(3.3) \qquad \lambda + e^{-\lambda r} = 0$$

There is a unique $r_0 = \pi/2$ such that Equation (3.3) has all roots with negative real parts for $0 < r < r_0$ and a unique

pair of purely imaginary roots $\lambda = \pm i$ for $r = r_0$. The curve Γ that interests us is the one in the (r_1, r_2)-plane which passes through the point $(\pi/2, \pi/2)$.

If $\lambda = i\gamma$ in Equation (3.2), then we must have

$$\begin{aligned} 0 &= \cos \frac{\gamma(r_1+r_2)}{2} \cos \frac{\gamma(r_1-r_2)}{2} \\ \gamma &= \sin \frac{\gamma(r_1+r_2)}{2} \cos \frac{\gamma(r_1-r_2)}{2} \end{aligned} \tag{3.4}$$

Since $\gamma = 0$ is not a solution of Equation (3.2), it follows that $\cos\ (r_1-r_2)/2 \neq 0$. Thus, we must have $\gamma = \pi/(r_1+r_2)$,

$$\frac{\pi}{r_1+r_2} = \cos \frac{\pi(r_1-r_2)}{2(r_1+r_2)}$$

Figure 4

The curve Γ is, therefore, the solution of this equation and is shown schematically in Figure 4.

Now consider a perturbation of Equation (3.1)

$$\dot{x}(t) = -\frac{1}{2}x(t-r_1) - \frac{1}{2}x(t-r_2) + f(x(t), x(t-r_1), x(t-r_2))$$

where $f(x,y,z)$ is continuous together with derivatives up through order three, f and its first partial derivatives vanish at $x = y = z = 0$. Theorem 2.1 implies there is a Hopf bifurcation at every point in Γ.

To obtain more information, one must assume something about the nonlinearity f. In particular, consider the equation

$$(3.5) \qquad \dot{x}(t) = -\frac{1}{2}x(t-r_1) - \frac{1}{2}x(t-r_2) + \beta x^3(t), \quad \beta \neq 0.$$

If we let $\alpha_0 = (r_1^0, r_2^0) \in \Gamma$, then the characteristic roots $\lambda(\alpha_0), \bar{\lambda}(\alpha_0)$ on the imaginary axis are given by $i\gamma_0, -i\gamma_0$, $\gamma_0 = \pi/(r_1^0 + r_2^0)$. Choose bases $\Phi(\alpha_0), \Psi(\alpha_0)$ as in the previous section for the decomposition of C as $C = P_{\alpha_0} \oplus Q_\alpha$ for the characteristic roots $\{i\gamma_0, -i\gamma_0\}$. In [22], it is shown that $p(a_0)$ in (2.28) is given by

$$(3.6) \qquad p(\alpha_0) = -kc(\alpha_0)\beta$$

where k is a positive constant,

$$(3.7) \qquad c(\alpha_0) = (1 - \gamma_0 \sin \gamma_0 r_2^0/2)$$

If $c(\alpha_0) \neq 0$, Theorem 2.2 implies there is a neighborhood $V \subset \mathbb{R}^2$ of $\alpha_0 \in \Gamma$ such that $V \backslash \Gamma = \Lambda_1 \cup \Lambda_2$ and

(i) there is no small nonconstant periodic solution of (3.4) for $(r_1, r_2) \in \Lambda_1$

(ii) there is a unique small nonconstant periodic solution of (3.4) for $(r_1, r_2) \in \Lambda_2$

that is, a small segment of Γ is a bifurcation curve.

Ruiz-Claeyssen [22] also showed that $c(\alpha_0)\beta > 0$ implies the periodic orbit is asymptotically stable and the bifurcation occurs as Γ is crossed from right to left. If $c(\alpha_0)\beta < 0$, the periodic orbit is unstable and the bifurcation occurs as Γ is crossed from left to right. There are some points α_0 on Γ where $c(\alpha_0) = 0$. No information is available near these points since higher order approximations in the bifurcation equations will be needed. However, we do see that a two-parameter problem is very different from a one parameter problem. The direction of bifurcation can change as we move along Γ. Stech [24] has also observed this phenomena in problems with two delays.

An obvious generalization of Equation (3.1) is

$$\dot{x}(t) = -(1-\alpha)x(t-r_1) - \alpha x(t-r_2), \quad 0 \leq \alpha \leq 1, \tag{3.8}$$

whose characteristic equation is

$$\lambda + (1-\alpha)e^{-\lambda r_1} + \alpha e^{-\lambda r_2} = 0. \tag{3.9}$$

The analysis of Equation (3.9) for all $\alpha \in [0,1]$, $r_1 > 0$, $r_2 > 0$ is difficult. From our point of view, we would like to know for a fixed α the curve Γ in (r_1, r_2)-space which passes through the point $(\pi/2, \pi/2)$ and has the property that $(r_1, r_2) \in \Gamma$ implies two roots purely imaginary and the remaining roots with negative real parts. This curve changes in a complicated way as a function of α and is not completely understood.

The equation for the purely imaginary roots $\lambda = i\gamma$ of Equation (3.9) are

$$\text{(a)} \quad (1-\alpha)\cos \gamma r_1 + \alpha \cos \gamma r_2 = 0$$

(3.10)

$$\text{(b)} \quad (1-\alpha)\sin \gamma r_1 + \alpha \sin \gamma r_2 = \gamma .$$

For $\alpha = 0$, the solutions of this equation are

$$\gamma r_1 = (2k+1)\pi/2, \ \gamma = (-1)^k, \ k = 0,1,2,\ldots,$$

and the special curve Γ is given by $r_1 = \pi/2$, r_2 arbitrary, and corresponds to $k = 0$, $\gamma = 1$.

For $\alpha = 1/2$, Equation (3.10) can be written as Equation (3.4) and has the solutions

(3.11)
$$\gamma = (2k+1)\pi/(r_1+r_2)$$
$$\frac{(2k+1)\pi}{r_1+r_2} = \cos(2k+1)\frac{\pi}{2} \ \frac{r_2-r_1}{r_2+r_1}, \quad k = 0,1,2,\ldots .$$

As remarked earlier, the special curve Γ corresponds to $k = 0$

and is shown schematically in Figure 4. To understand how Γ changes with α, it is necessary to investigate the solutions of Equation (3.11) for other values of k. For $k = 1$; that is, $\gamma = 3\pi/(r_1+r_2)$,

$$(3.12) \qquad \frac{3\pi}{r_1+r_2} \cos \frac{3\pi}{2} \frac{r_2-r_1}{r_2+r_1} .$$

The solutions of Equation (3.12) are depicted in Figure 5a for $r_2 \geq r_1$. In this figure, we have also included the original curve Γ.

The value of $\gamma = (2k+1)\pi/(r_1+r_2)$, $k = 0,1$, approaches zero as r_1, r_2 approach infinity along either of these curves; that is, for large values of $r_1 + r_2$, there are many purely imaginary roots of Equation (3.11) near zero. Consequently, for a small change in α from $\alpha = 1/2$ in Equation (3.10), it is conceivable that the curves representing the solutions of Equation (3.10) for large values of $r_1 + r_2$ can have a different topological structure than for $\alpha = 1/2$. This suspicion was verified on a computer by M. Michaud and T. Lewis for the values of α depicted in Figure 5 and corresponding to the two curves associated with $k = 0,1$ for $\alpha = 1/2$.

In Figure 5b, the smooth parabolic-like curve corresponds to the curve $k = 1$, $\gamma = 3\pi/(r_1+r_2)$ for $\alpha = 1/2$ and the other curve to $k = 0$, $\gamma = \pi/(r_1+r_2)$ for $\alpha = 1/2$. At the cusp point, there are two distinct pairs of purely imaginary roots on the imaginary axis. The number of roots with positive real parts

Figure 5a

Figure 5b

Figure 5c

Figure 5d

Figure 5e

near this cusp is indicated in Figure 5b. The Hopf bifurcation theorem can be applied along each smooth branch passing through this cusp. The end result is that it is possible to have the same number of periodic orbits as roots with positive real parts.

For $r_2 = 3r_1$, it is not difficult to show (see Stech [24]) that in a neighborhood of $\alpha = 1/4$, Equation (3.10) has one solution for $\alpha < 1/4$ and three solutions for $\alpha > 1/4$ with γr_1 near $\pi/2$. This is a different type of phenomena than the one exhibited near the cusp where the purely imaginary parts were different depending on which branch was followed. For α near $1/4$, the curve is smooth but has a "kink" in it (see Figure 5d). Near $\alpha = 1/4$, $r_2 = 3r_1$, it is possible to obtain three periodic orbits. Similar phenomena have been observed by Nussbaum [21].

For larger values of r_2 than shown in Figure 5, more "kinks" and cusps must appear because of the fact that Equation (5.10) are almost periodic functions in r_1, r_2. Therefore, similar bifurcation phenomena occur.

These results are important because they have the following implications. If the ratio of the delays is constant, say $r_2 = \beta r_1$, $\beta \geq 1$, then it is possible to have an α and numbers $1 < \beta_0 < \beta_1$ such that along the line $r_2 = \beta r_1$

(i) if $1 \leq \beta < \beta_0$ there is a unique intersection with Γ_α,

(ii) if $\beta_0 < \beta < \beta_1$, there are several intersections with Γ_α,

(iii) if $\beta_1 < \beta$, there is a unique intersection with Γ_α.

Thus, there may be more than one bifurcation of a nonlinear perturbation of Equation (3.8) as we move along a ray $r_2 = \beta r_1$.

For the particular problem

$$(3.13) \qquad \dot{x}(t) = -[(1-\alpha)x(t-r_1) + \alpha x(t-r_2)][1+x(t)]$$

$0 \leq \alpha \leq 1$, Nussbaum [21] has obtained some very interesting global results for r_1, r_2 not necessarily near the curve Γ_α but being in a wedge $r_2 = \beta r_1$, $1 \leq \beta \leq 3$. The technique is to use ejective fixed point theorems and he has proved the following results:

(i) if $1 \leq \beta \leq 2$, $r_2 = \beta r_1$ and r_1 is sufficiently large, there is a nonconstant periodic solution of Equation (3.11).

(ii) if $2 < \beta \leq 3$, $r_2 = \beta r_1$, there are some ranges of α such that for all r_1 sufficiently large there are at least two solutions of Equation (3.13).

Our results on local bifurcation theory will apply equally as well to the equation

$$(3.14) \qquad \dot{x}(t) = -\alpha \left[\int_{-1}^{-r_1} k(\theta)x(t+\theta)\,d\theta \right] [1+x(t)].$$

One can study the bifurcation as a function of α, r. One of the main difficulties always consists of analyzing the characteristic equation

$$\lambda + \alpha \int_{-1}^{-r} k(\theta) e^{\lambda\theta} d\theta = 0.$$

The existence of a bifurcation can be then determined. The specific nature of the bifurcation depends on the nonlinear terms. As remarked earlier, the most efficient scheme for treating the nonlinear terms near $x = 0$ is the method of averaging (see [5]).

Some global results for Equation (3.14) have been recently obtained by Walther [27] for $|r-1|$ small.

4. Infinite delays. In this section, we summarize part of the thesis of Lima [16] on the Hopf bifurcation theorem with infinite delays. In the case of finite delays, the solution x_t of a retarded equation will belong to C for $t \geq r$ regardless of the space of initial data for which one defines solutions. This is the reason why the initial space C or $\mathbb{R}^n \times L^p([-r,0],\mathbb{R}^n)$ or $W^{1,p}([-r,0],\mathbb{R}^n)$ has little effect on the qualitative theory. For infinite delays, the space of initial functions becomes more crucial since all of the past history is contained in x_t. In the first part of this section, we consider a special case of the initial space in Lima [16] and later make some remarks about other spaces.

Suppose $g, G: (-\infty,0] \to \mathbb{R}$, are nonnegative continuous functions satisfying

$$g(t+s) \leq G(t)g(s)$$

$$\int_{-\infty}^{0} g < \infty \tag{4.1}$$

there is a $\beta_0 \in (-\infty,0]$ such that $G(\beta_0) < 1$.

For example, if $g(\theta) = \exp \lambda\theta$, then $G(\theta) = \exp \lambda\theta$ and all hypotheses are satisfied if $\lambda > 0$. The function $g(\theta) = \theta^{-2}$ will not satisfy the hypotheses. Let

(4.2) B_2 = {equivalence classes of all measurable functions $\phi: (-\infty,0] \to \mathbb{R}^n$ such that $|\phi|_2 \overset{\text{def}}{=} \{|\phi(0)|^2 + \int_{-\infty}^{0} g(\theta)|\phi(\theta)|^2 d\theta\}^{1/2} < \infty\}$.

Suppose $\Omega \subset \mathbb{R}^k$, f, L: $\Omega \times B_2 \to \mathbb{R}^n$ are continuous, $L(\alpha)\phi$ is linear in ϕ, $f(\alpha,\phi)$ has continuous derivatives in ϕ up through order two, $f(\alpha,0) = 0$, $\partial f(\alpha,0)/\partial\phi = 0$. Consider the linear equation

$$\dot{x}(t) = L(\alpha)x_t \tag{4.3}$$

as well as the perturbed equation

$$\dot{x}(t) = L(\alpha)x_t + f(\alpha,x_t) \tag{4.4}$$

Since $L(\alpha)\phi$ is linear in ϕ, it can be written as

$$L(\alpha)\phi = A(\alpha)\phi(0) + \int_{-\infty}^{0} k(\alpha,\theta)\phi(\theta)g(\theta)d\theta \tag{4.5}$$

where $k(\alpha,\cdot) \in L^2[(-\infty,0),\mathbb{R}^n]$ and $A(\alpha)$ is an $n \times n$ constant matrix. In the remainder of the section, we assume $L(\alpha)\phi$ is given by Relation (4.5) and that $A(\alpha)$, $k(\alpha,\cdot)$ are continuous in α. The characteristic equation for Equation (4.3) is

$$\det \Delta(\alpha,\lambda) = 0 \tag{4.6}$$
$$\Delta(\alpha,\lambda) = \lambda I - A(\alpha) - \int_{-\infty}^{0} k(\alpha,\theta) e^{\lambda\theta} g(\theta)d\theta.$$

The formal adjoint equation is

$$\dot{y}(t) = -y(0)A(\alpha) - \int_{-\infty}^{0} y(t-\theta)k(\alpha,\theta)g(\theta)d\theta \tag{4.7}$$

and the corresponding bilinear form is

$$<\psi,\phi> = \psi(0)\phi(0) + \int_{-\infty}^{0} \left[\int_{\theta}^{0} \psi(\theta-u)k(\alpha,\theta)\phi(u)du\right] g(\theta)d\theta \tag{4.8}$$

defined for $\phi \varepsilon B_2$ and an n-dimensional row vector function ψ in some appropriate class of functions. The specific class is not important to us here and may be found in [16]. It is only necessary to remark that $<\psi,\phi>$ will be meaningful for all functions considered below.

Before stating the Hopf bifurcation theorem, let us point out the important implications of $G(\beta_0) < 1$ in Relation (4.1). If $x(\phi)$ is the solution of Equation (4.3) for $\phi \varepsilon B_2$ and $T(t)\phi \overset{def}{=} x_t(\phi)$, $t \geq 0$, then $\{T(t), t \geq 0\}$ is a strongly continuous semigroup of linear operators on B_2. If $G(\beta_0) < 1$, then there is a $\mu > 0$ such that the essential spectrum of $T(t)$ belongs to the circle of radius $e^{-\mu t}$, $t \geq 0$ (see [16], [19] or [10]). Therefore, only elements of the point spectrum of $T(t)$ lie outside this circle and these elements of the point spectrum are $e^{-\lambda t}$ where λ satisfies

the Characteristic Equation (4.6). This makes the Hopf bifurcation theorem feasible since one can have two characteristic values on the imaginary axis and know that it is possible to have the other elements of the spectrum of $T(1)$ inside the unit circle.

The discussion below is a slight generalization of Lima [16] taking into account the fact that we know from Section 2 that one can prove bifurcation theorems without having $L(\alpha)\phi, f(\alpha,\phi)$ differentiable in α for all functions if we are very careful in the analysis.

The hypotheses imposed for a bifurcation theorem are the same as the ones in Section 2 for finite delay.

(H_1) <u>The characteristic matrix</u> $\Delta(\alpha,\lambda)$ <u>in Relation</u> (4.6) <u>is continuously differentiable in</u> α, <u>there is a purely imaginary characteristic root</u> $\lambda_0 = i\nu_0, \nu_0 > 0$, <u>for</u> $\alpha = \alpha_0$ <u>and no other characteristic root</u> $\lambda_j \neq \lambda_0, \bar{\lambda}_0$ <u>for</u> $\alpha = \alpha_0$ <u>satisfies</u> $\lambda_j = m\lambda_0$, m <u>an integer</u>;

(H_2) <u>for any</u> $K > 0$; $\phi: (-\infty, 0] \to \mathbb{R}^n$ <u>uniformly continuous together with its first derivative bounded by</u> K, <u>the function</u> $L(\alpha)\phi + f(\alpha,\phi)$ <u>has a derivative in</u> α <u>which is continuous in</u> α,ϕ.

(H_3) $\zeta(\alpha_0) \neq 0$.

The number $\zeta(\alpha_0)$ is defined in the following manner.

$\lambda_0 = i\nu_0, \bar{\lambda}_0 = -i\nu_0$ are characteristic roots of Equation (4.3) for $\alpha = \alpha_0$. There are characteristic roots $\lambda(\alpha), \bar{\lambda}(\alpha)$ of Equation (4.3) which are continuously differentiable in α, $\lambda(\alpha_0) = i\nu_0$, and two linear independent solutions $b \exp \lambda(\alpha)t$, $\bar{b} \exp \bar{\lambda}(\alpha)t$ of Equation (4.3). Defining

$$\Phi_\alpha = (\phi_1, \phi_2), \quad \phi_1(\theta) = \mathrm{Re}\, b \exp \lambda(\alpha)\theta,$$
$$\phi_2(\theta) = \mathrm{Im}\, b \exp \lambda(\alpha)\theta,$$

then we may assume (by a change of basis if necessary) that

$$\Phi_\alpha(\theta) = \Phi_\alpha(0)e^{B(\alpha)\theta}, \qquad -\infty < \theta \leq 0$$

$$B(\alpha) = \nu_0 B_0 + B_1(\alpha)$$

$$B_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_1(\alpha) = \begin{bmatrix} (\alpha-\alpha_0)\cdot\zeta(\alpha) & (\alpha-\alpha_0)\cdot\gamma(\alpha) \\ -(\alpha-\alpha_0)\cdot\gamma(\alpha) & (\alpha-\alpha_0)\cdot\zeta(\alpha) \end{bmatrix}.$$

Under hypotheses (H_1)-(H_3) the Bifurcation Theorem 2.1 holds. The proof follows the one given in [16] under less general hypotheses. The appropriate generalization of Theorem 2.2 is also valid.

In some applications, it may be more appropriate to have other spaces of initial data. For any $p \geq 1$, one can consider B_p defined in Relation (4.2) with 2 replaced by p (see [16]). Another interesting space is the following. For any real number γ, let

$$(4.9)\qquad C_\gamma = \{\phi : (-\infty,0] \to \mathbb{R}^n \text{ continuous, } e^{-\gamma\theta}\phi(\theta) \to \text{a limit as } \theta \to -\infty\}$$

with $|\phi|_\gamma = \sup_\theta\{|\phi(\theta)|\exp - \gamma\theta\}$. If $\gamma > 0$, the essential spectrum of the semigroup $T(t)$ has the same properties as in B_2 (see [10]). The Hopf bifurcation should, therefore, be valid for C_γ, $\gamma > 0$. A recent preprint of Naito [20] indicates it will be possible to supply a proof of this result in a manner similar to the one given in Section 2. Also, this preprint suggests that the same will be true for the more general spaces of initial data considered by Hale and Kato [10].

Considering the example in [16], suppose k is nonnegative, continuous on $(-\infty,0]$, $\int_{-\infty}^{0} k = 1$ and consider the scalar equation

$$(4.10)\qquad \dot{x}(t) = -\alpha \int_{-\infty}^{-r} k(\theta+r)f(x(t+\theta))d\theta$$

where $r \geq 0$, $\alpha \geq 0$ are constants, f is continuous together with derivatives up through order two, $f(0) = 0$, $f'(0) = 1$. We wish to study the Hopf bifurcation for Equation (4.9) as a function of (α,r).

In [12], Israelson and Johnson and in [13], Johnson and Karlsson proposed the Equation (4.10) as a mathematical model to study the rhythmic movement of some plants. Experimental evidence indicates the kernel k should have the form shown

in Figure 6. The particular form of k is not important below.

Figure 6

The first problem is to choose the space of initial functions so that the previous theory applies. If there is a $\rho > 0$ and constant K such that $|k(\theta)| \leq K \exp \rho\theta$, for $-\infty < \theta \leq 0$, then we take $g(\theta) = \exp \mu\theta$, $0 < \mu < \rho$ and define

$$B_2 = \{\phi: (-\infty,0] \to \mathbb{R}: \phi \text{ is locally integrable,}$$

$$|\phi|^2 = |\phi(0)|^2 + \int_{-\infty}^{0} e^{\mu\theta}|\phi(\theta)|^2 d\theta < \infty\}$$

In order for the right hand side of Equation (4.10),

$$F(r,\alpha,\phi) \overset{\text{def}}{=} -\alpha \int_{-\infty}^{-r} k(\theta+r)f(\phi(\theta))d\theta \tag{4.11}$$

to be continuous in (r,α,ϕ) and continuously differentiable in α,ϕ, we suppose f satisfies

$$|f(x)| \leq a + bx^2, \quad |f'(x)| \leq a_1 + b_1|x|$$

for all x and some constants a,b,a_1,b_1 (see [16] or [26]). The function $F(r,\alpha,\phi)$ is not continuously differentiable in r but it does satisfy Hypothesis (H_2) above.

The linearization of Equation (4.9) around $x = 0$ is

$$\dot{x}(t) = -\alpha \int_{-\infty}^{-r} k(\theta+r)x(t+\theta)d\theta. \tag{4.12}$$

The characteristic equation is

$$\lambda + \alpha \int_{-\infty}^{-r} k(\theta+r)e^{\lambda\theta}d\theta = 0. \tag{4.13}$$

Stech [23] has shown that the Equation (4.13) has all roots with negative real parts if

$$\alpha \int_{-\infty}^{0} \theta k(\theta)d\theta > -1. \tag{4.14}$$

Therefore, no bifurcation can occur if Relation (4.14) is satisfied.

If $r > 0$, $\tilde{\lambda} = r\lambda$, $\tilde{\alpha} = r\alpha$, then Equation (4.12) is equivalent to

$$\tilde{\lambda} + \tilde{\alpha}e^{-\tilde{\lambda}} \int_{-\infty}^{-1} e^{\lambda\theta} k(r\theta)d(r\theta) = 0. \tag{4.15}$$

Stech [23] has shown that for $\tilde{\alpha} > \pi/2$ there is an $r_0 > 0$ such that Equation (4.15) has roots with positive real parts. This certainly suggests there must be a Hopf bifurcation for Equation (4.19) for some value of $\tilde{\alpha}, r$. To show that a smooth bifurcation occurs at some point, one must be able to say something about the derivatives of the roots as they cross the imaginary axis, generally a very difficult problem. Lima [16] has discussed some particular cases for the function k to show that smooth bifurcations do occur. In the class of all continuous kernels k one can say without any computations that generically there will always be smooth bifurcations.

Of course, once the linear equation (4.13) has been analyzed, the existence of a Hopf bifurcation will also be true for the equation

$$\dot{x}(t) = -\alpha\left[\int_{-\infty}^{-r} k(\theta+r)x(t+\theta)\right][1+x(t)]$$

as well as more general equations.

It is possible also to consider the linear equation to have the form

$$\dot{x}(t) = -\alpha \int_{-\infty}^{-r} x(t+\theta)dK(\theta+r) \tag{4.16}$$

where $K(\theta)$ is a nondecreasing function on $(-\infty,0]$. This allows discrete delays as well as integral dependence over the past history. If we suppose there is a $\mu > 0$ such that

$$\int_{-\infty}^{0} e^{-\mu\theta} dK(\theta) < \infty$$

then the previous theory will be applicable by choosing the space B_2. Of course, there must be a few technical changes to take care of the fact that $\phi(-1)$ for example is not a function on integrable functions on $(-\infty,0]$. A more convenient space would probably be C_μ defined in Relation (4.9).

The results of Stech [23] are also valid for Equation (4.16). Therefore, bifurcation problems can be considered for either

$$\dot{x}(t) = -\alpha \int_{-\infty}^{-r} f(x(t+\theta))dK(\theta+r)$$

$$\dot{x}(t) = -\alpha \left[\int_{-\infty}^{-r} x(t+\theta)dK(\theta+r)\right][1+x(t)]$$

or more general equations.

5. Stabilizing effect of delays.. In many applications of delay equations, an increase in the delay tends to have a destabilizing effect (see, for example, Cushing [6], May [18]). In population dynamics, it has been noted that it can have a stabilizing effect. The rate of growth away from an unstable equilibrium can be decreased by an increase in the delay (see, for example, Cushing [6, p. 35ff]). For some population models representing two species, Cushing [6, p. 80ff] proves that an unstable equilibrium cannot be made stable by increasing the delay although the instability can be weakened in the above sense. He also gives an example [6, p. 99ff] in three dimensions for which an unstable equilibrium can be made stable by increasing the delays. This means stable Hopf bifurcation can occur by decreasing the delays.

It is possible also to give an example in two dimensions where unstable equilibrium can be made stable by increasing the delays. The purpose of this section is to give an example of this type.

Consider the system

$$\begin{aligned} \dot{y}(t) &= -c_{11}y(t) + c_{12}z(t) + b_1 y(t-r_1) \\ \dot{z}(t) &= c_{21}y(t) - c_{22}z(t) + b_2 y(t-r_2) \end{aligned} \tag{5.1}$$

where $c_{ij} > 0$, $b_j > 0$, $r_j > 0$ are constants. No claim is made to the physical importance of this equation, but it is

similar to the symbiotic case in population dynamics (see Keyfitz [14, p. 273]).

If $C = (c_{ij})$, we suppose

$$(c_{11} + c_{22})^2 - 4 \det C < 0. \tag{5.2}$$

This implies the eigenvalues of C are $-a_1 \pm ia_2$ where $a_1 > 0$, $a_2 > 0$; that is, the eigenvalues of C have negative real parts and are complex conjugate. To simplify computations, also suppose

$$b_1 = b_2 = b, \quad r_1 = r_2 = r.$$

By a change of coordinates

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = D \begin{pmatrix} y \\ z \end{pmatrix}$$

we may assume System (5.1) has the form

$$\begin{aligned} &\dot{x}(t) = Ax(t) + bx(t-r) \\ &A = \begin{pmatrix} -a_1 & -a_2 \\ a_2 & -a_1 \end{pmatrix} \end{aligned} \tag{5.3}$$

If $-a_1 + b > 0$, $r = 0$, the ordinary differential (5.3) has eigenvalues with real parts positive and, thus, $x = 0$ is

unstable. It has been shown by Tsen [25] that for $a_2 r = \pi/2$ there is an $r_0 > 0$ such that the zero solution of Equation (5.3) is asymptotically stable for $r > r_0$. We reproduce some of these computations here.

By another change of coordinates, one can reduce Equation (5.3) to a scalar equation in a single complex variable w,

$$\dot{w} = (-a_1 + ia_2)w + bw(t-r), \tag{5.4}$$

for which the characteristic equation is

$$\lambda = -a_1 + ia_2 + be^{-\lambda r} \tag{5.5}$$

If $\lambda = \alpha + i\beta$, α, β, real, then

$$\begin{aligned} \alpha &= -a_1 + be^{-\alpha r}\cos \beta r \\ \beta &= \ a_2 - be^{-\alpha r}\sin \beta r \end{aligned} \tag{5.6}$$

or

$$\begin{aligned} \beta &= a_2 \pm [b^2 e^{-2\alpha r} - (\alpha + a_1)^2]^{1/2} \\ \alpha &= -a_1 + be^{-\alpha r}\cos \beta r. \end{aligned} \tag{5.7}$$

Any real solution of these equations must satisfy $0 \le \alpha \le \alpha_0(r)$ where $\alpha_0(r)$ is the unique positive zero of

$$f(\alpha,r) = b^2 e^{-2\alpha r} - (\alpha+a_1)^2. \tag{5.8}$$

For $0 \leq \alpha \leq \alpha_0(r)$, $f(\alpha,r) \geq 0$ and $f(\alpha,r)r \to 0$ as $r \to \infty$ uniformly in α. Choose $\varepsilon > 0$ such that $|b|\cos(\pi/2+\varepsilon) < a_1$ and choose $r_0 > 0$ such that $|f(\alpha,r)r| < \varepsilon$ for $0 \leq \alpha \leq \alpha_0(r)$, $r > r_0$. Define

$$\beta = a_2 + [b^2 e^{-2\alpha r} - (\alpha+a_1)^2]^{1/2}.$$

Since $a_2 r = \pi/2$, we have $|\beta r - \pi/2| < \varepsilon$ and

$$\begin{aligned} \alpha &= -a_1 + be^{-\alpha}\cos \beta r \\ &< -a_1(1-e^{-\alpha}) \end{aligned}$$

for $r > r_0$. This latter inequality implies $\alpha < 0$ since $a_1 > 0$. This proves the assertion that increasing the delay stabilizes Equation (5.1) for $r_1 = r_2 = r$, $a_2 r = \pi/2$, $b_1 = b_2$.

As remarked earlier, this implies that Equation (5.1) under the above hypotheses and when subjected to nonlinear perturbations which vanish together with their first derivatives, will generally have a stable Hopf bifurcation occur as the delay decreases. We start with $r > r_0$ where the zero solution is stable and decrease r.

6. Nonautonomous equations. The previous discussion is concerned only with the case when the vector field in the equations is independent of time - the autonomous case. When the equation is nonautonomous, the problem is much more difficult. The basic difficulty is not due to the delays, but is a consequence of the fact that it is difficult to determine the effect of nonlinearities in resonance phenomena. The best procedure available for discussing this problem locally is general transformation theory, averaging methods and the method of integral manifolds. For ordinary differential equations, the basic ideas in the transformation theory go back as far as Liapunov [15] (and perhaps further) when he was concerned with the problem of determining conditions for the stability of an equilibrium point for a nonlinear system when the coefficient matrix of the linear part has zero roots or purely imaginary roots. Certain aspects of the method of averaging were also encountered by Liapunov. The more general theory of averaging as well as the theory of integral manifolds was discovered by Krylov and Bogoliubov in the 1930's (see Bogoliubov and Mitropolski [2] or Hale [8]). The transformation theory is extensive (see, for example, Malkin [17], Brjuno [3]).

The generalization of these ideas to delay equations of retarded and neutral type began in the late 1950's. An idea of the contributions in this development can be found in [17]. At this time, we can safely say that most of the results that are known in this area for ordinary differential equations are

known for delay equations or the necessary machinery is available to obtain them.

It is not feasible to present the elements of this theory in a few pages - a book would be more reasonable. Therefore, we are not going to discuss delay equations. Furthermore, we discuss only one problem for a very simple ordinary differential equation. To avoid any technicalities, we also assume all functions are very smooth in all variables.

Consider the second order equation

$$\dot{x} = Bx + f(t,x,\varepsilon) \tag{6.1}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $f = (f_1, f_2) \in \mathbb{R}^2$, B is a 2×2 constant matrix, the real parts of the eigenvalues of B are zero,

$$B = \begin{bmatrix} 0 & \sigma \\ -\sigma & 0 \end{bmatrix}, \quad \sigma > 0 \tag{6.2}$$

the function $f(t,x,\varepsilon)$ is T-periodic in t,

$$\begin{aligned} f(t,0,\varepsilon) &= 0 \\ f(t,x,0) &= O(|x|^2) \quad \text{as} \quad |x| \to 0. \end{aligned} \tag{6.3}$$

Hypothesis (6.3) implies $x = 0$ is a solution of Equation (6.1). Our objective is to discuss the behavior of

the solutions of Equation (6.1) near $x = 0$. In the previous sections, we have considered this problem for $f(t,x,\varepsilon)$ independent of t and observed that it was possible for a Hopf bifurcation to occur at $\varepsilon = 0$. The bifurcating periodic orbit gave a closed curve Γ in $\mathbb{R}^2$ and a cylinder $\Gamma \times \mathbb{R}$ of solutions in (x,t)-space in $\mathbb{R}^2 \times \mathbb{R}$. The cylinder is obtained because to any nonconstant periodic solution of an autonomous equation, one obtains another solution by a phase shift (see Figure 7)

Figure 7

If the bifurcating periodic orbit is asymptotically stable, then the cylinder $\Gamma \times \mathbb{R}$ is asymptotically stable in $\mathbb{R}^2 \times \mathbb{R}$. If the autonomous differential equation is subjected to small periodic disturbances, we do not expect to preserve the same kind of orbit structure as before (families of periodic solutions, for example), but we might expect to preserve a surface similar to the cylinder $\Gamma \times \mathbb{R}$ which would be asymptotically stable and completely filled up with solutions of the equation. A surface with this latter property is called an integral manifold.

With the above motivation, let us take as our objective the determination of integral manifolds in a neighborhood of the set $\{(x,t): x = 0, t \in \mathbb{R}\}$ which are like cylinders. More specifically, we wish to find a function $g: \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$, $g(t,\theta)$ periodic in θ of period $2\pi/\sigma$, periodic in t of period T such that the set

$$S = \{(x,t) \in \mathbb{R}^2 \times \mathbb{R}: x = g(t,\theta),\ (t,\theta) \in \mathbb{R}^2\}$$

is an integral manifold; that is if $(\xi,\sigma) \in S$, then the solution $x(t)$ of Equation 6.1) through (ξ,σ) exists for all $t \in \mathbb{R}$ and $(x(t),t) \in S$ for $t \in \mathbb{R}$.

To obtain such integral manifolds, let

$$x_1 = \rho \sin \sigma\theta, \quad x_2 = \rho \cos \sigma\theta \tag{6.4}$$

to obtain

$$(6.5)\qquad \begin{aligned} \dot{\theta} &= 1 + \Xi(t,\theta,\rho,\varepsilon) \\ \dot{\rho} &= R(t,\theta,\rho,\varepsilon) \end{aligned}$$

where

$$(6.6)\qquad \begin{aligned} \Xi(t,\theta,\rho,\varepsilon) &= (\rho\sigma)^{-1}[f_1\cos\sigma\theta - f_2\sin\sigma\theta] \\ R(t,\theta,\rho,\varepsilon) &= f_1\sin\sigma\theta + f_2\cos\sigma\theta \end{aligned}$$

and the arguments of f_1, f_2 are (t,x,ε) with x given in Relation (6.4). The functions Ξ, R satisfy

$$(6.7)\qquad \begin{aligned} \Xi &= O(|\rho|+|\varepsilon|) \\ R(t,0,\varepsilon) &= 0 \\ R(t,\rho,0) &= O(|\rho|^2), \end{aligned}$$

are T-period in t, $2\pi/\sigma$-periodic in θ.

We do not attempt to analyze Equation (6.5) completely but only give a special result to illustrate some of the basic ideas. Let

$$(6.8)\qquad \begin{aligned} \Xi(t,\theta,\rho,\varepsilon) &= 1 + \alpha_1(t,\theta,\varepsilon) + \alpha_2(t,\theta,\varepsilon)\rho + \alpha_3(t,\theta,\varepsilon)\rho^2 + O(|\rho|^3) \\ R(t,\theta,\rho,\varepsilon) &= \beta_1(t,\theta,\varepsilon)\rho + \beta_2(t,\theta,\varepsilon)\rho^2 + \beta_3(t,\theta,\varepsilon)\rho^3 + O(|\rho|^4). \end{aligned}$$

The objective is to transform to new coordinates

$$\xi = \theta + u_0(t,\theta,\varepsilon) + u_1(t,\theta,\varepsilon)\rho + u_2(t,\theta,\varepsilon)\rho^2 + O(|\rho|^3)$$

(6.9)

$$r = \rho + v_1(t,\theta,\varepsilon)\rho + v_2(t,\theta,\varepsilon)\rho^2 + O(|\rho|^3)$$

where all functions are T-periodic in t and $2\pi/\sigma$ periodic in t in such a way that the new equations in ζ, r have the form

$$\dot{\zeta} = 1 + \mu_1(\varepsilon) + \mu_2(\varepsilon)r + \mu_3(\varepsilon)r^2 + O(|r|^3)$$

(6.10)

$$\dot{r} = \nu_1(\varepsilon)r + \nu_2(\varepsilon)r^2 + \nu_3(\varepsilon)r^3 + O(|r|^4)$$

If one formally substitutes Relations (6.9) in Equation (6.5) and equates coefficients of r, then one observes that each of the functions u_j, v_j satisfy a partial differential equation of the type

(6.11) $$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial \theta} = h(t,\theta)$$

where $h(t,\theta)$ is T-periodic in t and $2\pi/\sigma$ periodic in θ. To obtain a solution of Equation (6.11) which is T-periodic in t and $2\pi/\sigma$ periodic in θ, and must impose some restrictions on ω, σ, where $\omega = 2\pi/T$. In fact, if

$$u(t,\theta) = \sum u_{k\ell} e^{i(k\omega t+\ell\sigma\theta)}$$

$$h(t,\theta) = \sum h_{k\ell} e^{i(k\omega t+\ell\sigma\theta)}$$

then we can uniquely determine $u_{k\ell}$ as

$$u_{k\ell} = [i(k\ \ +\ell\sigma)]^{-1} h_{k\ell}$$

is $k\omega + \ell\sigma \neq 0$ for every $h_{k\ell}$ that is not zero. If $h_k = 0$ we take $u_{k\ell} = 0$. Therefore, if we assume $k\omega + \ell\sigma \neq 0$ for those $k \neq 0$, $\ell \neq 0$ which occur in the Fourier series for h, the function u will be uniquely determined if

$$(6.12) \qquad \int_0^{2\pi/\sigma} \int_0^{2\pi/\omega} h(t,\theta)\,dt\,d\theta = 0.$$

In general, Relation (6.12) will not be satisfied. The coefficients $\mu_j(\varepsilon)$, $\nu_j(\varepsilon)$ are chosen so that h will satisfy Relation (6.12). Thus, $\mu_j(\varepsilon), \nu_j(\varepsilon)$ are the mean values of some functions explicitly obtained in the process of computing the u_j and v_j. For more details on methods of computation, the reader is referred to [21], [51], [17].

If we assume the nonresonance conditions are satisfied for ω, σ and Equation (6.5) has been transformed to Equation (6.10), then we can state the following result which is a consequence of the theory of integral manifolds.

Theorem 6.1. *If*

$$(6.13)\qquad \begin{aligned} &\nu_1'(0) \neq 0 \\ &\nu_2(0) = \nu_2'(0) = 0 \\ &\nu_3(0) \neq 0, \qquad \nu_3(0)\nu_1'(0) < 0 \end{aligned}$$

then there is an $\varepsilon_0 > 0$ *and a function* $g(t,\theta,\mu)$, $0 \le \mu < \sqrt{\varepsilon}_0$, *T-periodic in* t, $2\pi/\sigma$*-periodic in* ω, $g(t,\theta,0) = [-\nu_3(0)\nu_1'(0)]^{1/2}$ *such that the set*

$$(6.14)\qquad S_\varepsilon = \{(x,t) \in \mathbb{R}^2 \times \mathbb{R};\ x_1 = \sqrt{\varepsilon}\, g(t,\theta,\sqrt{\varepsilon})\sin\sigma\theta,$$
$$x_2 = \sqrt{\varepsilon}\, g(t,\theta,\sqrt{\varepsilon})\cos\sigma\theta, \quad \theta \in \mathbb{R},\ t \in \mathbb{R}\}$$

$0 \le \varepsilon < \varepsilon_0$ *is an integral manifold of Equation* (6.1). *Furthermore, this integral manifold is asymptotically stable if* $\nu_1'(0) > 0$ *and unstable if* $\nu_1'(0) < 0$.

If $\nu_3(0)\nu_1'(0) > 0$, *the same conclusions hold on existence if* $-\varepsilon_0 < \varepsilon \le 0$ *and* S_ε *is unstable if* $\nu_1'(0) < 0$ *and asymptotically stable if* $\nu_1'(0) > 0$.

Let us make a few more remarks about second order ordinary differential equations. Consider the equation

$$(6.15)\qquad \dot{y} = By + F(t,y,\varepsilon)$$

where B is the same as before, F is T-periodic in t,

$$F(t,0,0) = 0$$
$$F(t,y,0) = O(|y|^2).$$

Assume the linear equation

$$\dot{y} = By + h(t)$$

for a T-periodic function h has a unique T-periodic solution. Then the nonlinear Equation (6.16) will have a unique T-periodic solution $y^*(t,\varepsilon)$ in a neighborhood of $x = 0$, $\varepsilon = 0$, $y^*(t,0) = 0$ (see, for example, [8]). If

$$y = y^*(t,\varepsilon) + x$$

then x satisfies an equation of the same form as Equation (6.1). Thus, the above result applies to Equation (6.16).

For equations with delays, results similar to Theorem 6.1 can be proved. However, much more machinery is needed. One must use the decomposition theory of linear autonomous equations (see Hale [7, Ch. 7, 9]), averaging methods (see Chow and Mallet-Paret [5] or Hausrath [11]) and the generalization of the methods of integral manifolds to delay equations (see, for example, Hale [7, Ch. 10, Sec. 10.4] for some references). If one is very familiar with these methods in ordinary differential equations, the ideas used in delay equations are not too surprising. Before tackling these problems for delay equations,

I would suggest that the reader be able to supply all details in the previous example.

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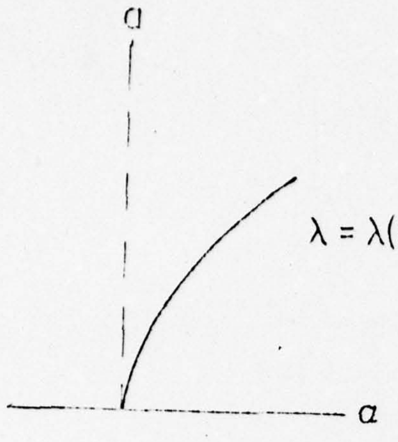

FIG. 1

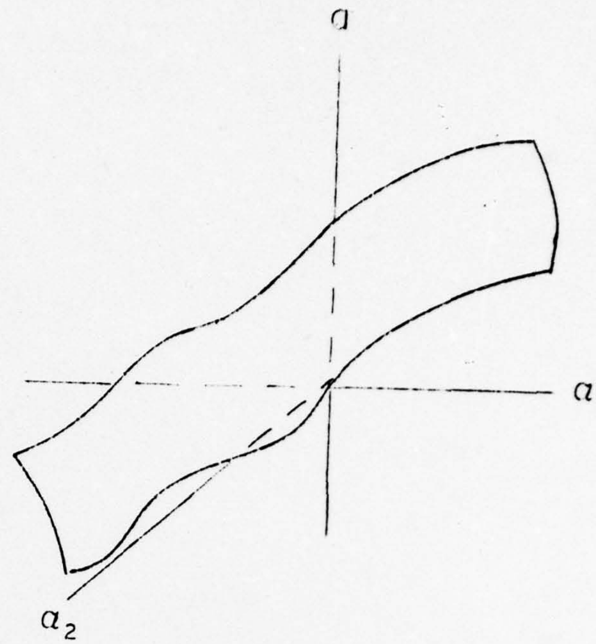

FIG. 2

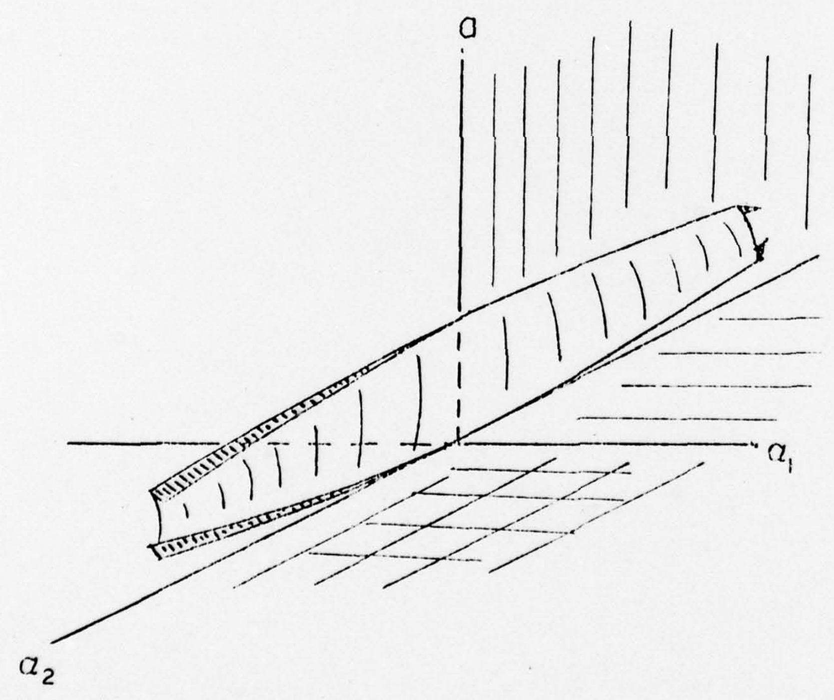

FIG. 3.

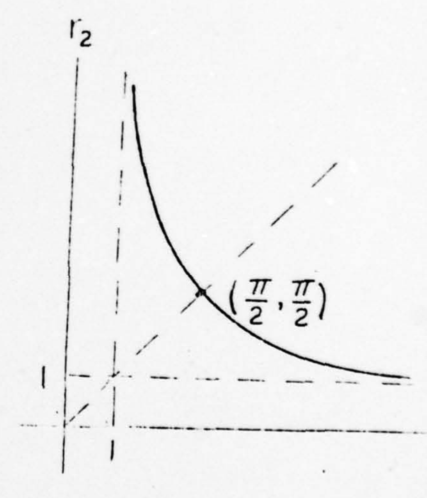

FIG. 4.

FIG. 5a ($\alpha = 0.5$)

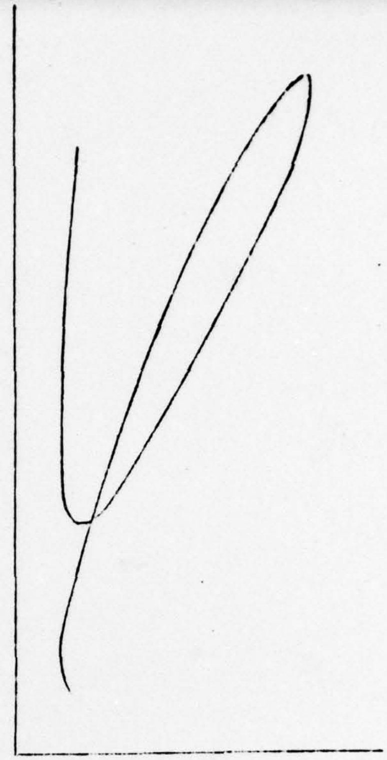

FIG. 5b ($\alpha = 0.4$)

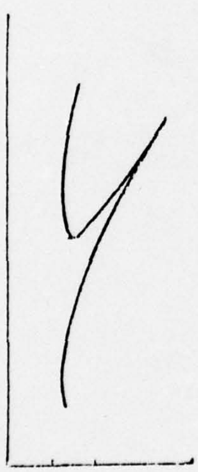

FIG. 5c ($\alpha = 0.3$)

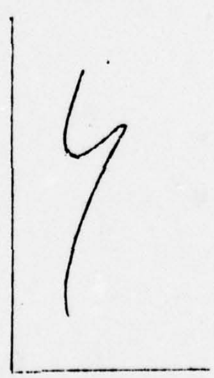

FIG. 5d ($\alpha = 0.2$)

FIG. 5e ($\alpha = 0.1$)

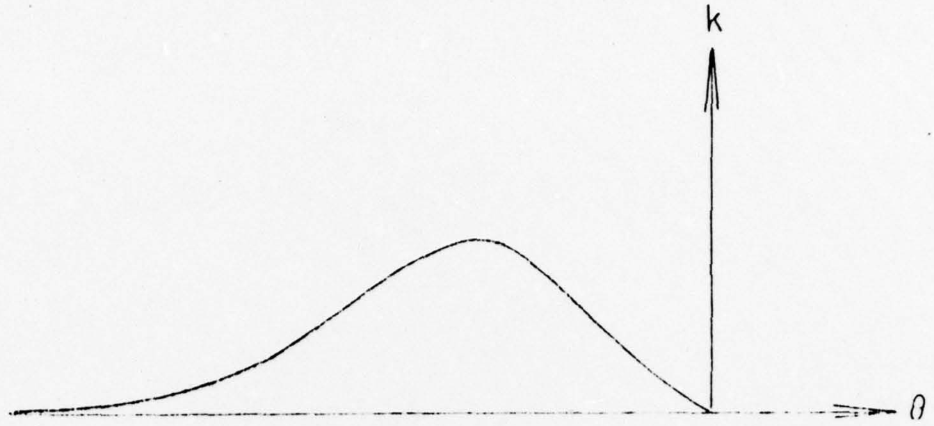

FIG. 6

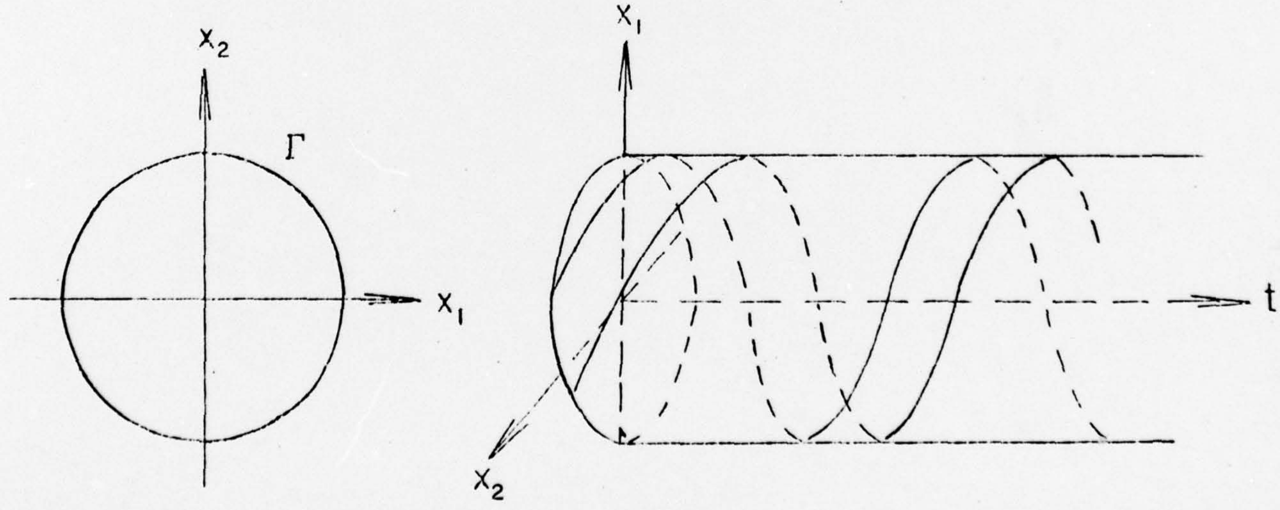

FIG. 7

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